International Journal of Theoretical Physics, Vol. 45, No. 5, May 2006 (© 2006)
DOI: 10.1007/s10773-006-9092-5

Radiation Characteristics of Stationary Axisymmetric Sen Black Hole as Tunneling

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Received June 27, 2005; accepted January 26, 2006
Published Online: April 11, 2006

Taking energy conservation and angular momentum conservation into account, the tunneling radiation characteristics of stationary axisymmetric Sen black hole is studied in this paper with the quantum tunneling method and the results show that the tunneling rate of particle at the event horizon of the black hole is relevant to Bekenstein–Hawking entropy and that the radiation spectrum is not strictly pure thermal.

KEY WORDS: Sen black hole; energy conservation; angular momentum conservation; tunneling rate.
PACS: 04.70_S, 97.60.Lf

1. INTRODUCTION

At the beginning of the 1970s, Hawking discovered and proved the thermal radiation of black hole (Hawking, 1975), which is greatly meaningful to research the evolution of fixed stars. Since then, black hole's thermodynamic have been studied by lots of people. There are two methods to research Hawking radiation, namely quantum field theory method and Damour–Ruffini method. The common factors between these two is that the background space-time of black hole is fixed and that the obtained thermal spectrum is pure thermal (Liu and Xu, 2002; Zhao *et al*., 2000; Yang, 2004). In fact, the mass varies with the black hole radiation, accordingly the position of the event horizon of black hole also varies so much so that the energy conservation should be considered during radiation. Recently, Parikh put forward a tunneling model and carried out a research on spherically symmetric black holes like Schwarzschild and Reissner–Nordström, under the account that the radiation spectrum after that black hole emission particles bring about the variation of the event horizon and the energy of black hole. The research results display that the radiation spectrum is not pure thermal and then the

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spherical-symmetric black hole radiation is added some new amendments, which extends our understanding on quantum tunnel effect (Parikh, 2004; Parikh and Wiltzek, 2000; Parikh, 2004). Stationary axisymmetric Sen black hole describes a solution of revolving charges dilation black hole, which owns mass, charge, angular momentum and magnetic monopoles. Due to the angular speed of the Sen black hole $\Omega \neq 0$ and magnetic monopoles $\mu \neq 0$, the discussion in this paper differs from Parikh's. The quantum tunneling radiation characteristics of such stationary black hole is studied and the results are significantly important for further study. The outline of our paper is as follows. We study the event horizon and infinite red-shift surface of the stationary axisymmetric Sen black hole in Section 2; In Section 3, we make general Painleve coordinate transformation to space-time line element of Sen black hole under dragging coordinate system; In Section 4, we analyze the tunneling characteristics of radiation at the event horizon of the black hole; At last, we make discussion on the tunneling characteristics.

2. THE EVENT HORIZON AND INFINITE RED-SHIFT SURFACE OF STATIONARY AXISYMMETRIC SEN BLACK HOLE

The space-time line element of stationary axisymmetric Sen black hole is (Sen, 1992)

$$ds^2 = -\frac{\sum - a^2 \sin^2\theta}{\Delta} dt_s^2 + \frac{\Delta}{\sum} dr^2 + \Delta d\theta^2 + \frac{\sin^2\theta}{\Delta}\Big[(r^2 + a^2 - 2\beta r)^2 - \sum a^2 \sin^2\theta\Big] d\varphi^2 - \frac{2a\sin^2\theta}{\Delta}\Big[(r^2 + a^2 - 2\beta r) - \sum\Big] dt_s d\varphi, \quad (1)$$

in which t_s represents the time coordinate of the Sen black hole

$$\sum = r^2 - 2mr + a^2, \quad \Delta = r^2 - 2\beta r + a^2\cos^2\theta, \quad \beta = -m\,\mathrm{sh}^2\frac{\alpha}{2}, \quad (2)$$

in which α is a parameter and the metric describes the black hole with mass M, charge Q, angular momentum J and magnetic monopoles μ. The variables are

$$M = \frac{1}{2}(1 + \mathrm{ch}\alpha) = m\,\mathrm{ch}^2\frac{\alpha}{2}, \quad Q = \frac{m}{\sqrt{2}}\mathrm{sh}\alpha = \sqrt{2}m\,\mathrm{sh}\alpha\,\mathrm{ch}\frac{\alpha}{2},$$

$$J = \frac{m\alpha}{2}(1 + \mathrm{ch}\alpha) = m\alpha\,\mathrm{ch}^2\frac{\alpha}{2}, \quad \mu = \frac{1}{\sqrt{2}}\mathrm{m}a\,\mathrm{sh}\alpha, \quad (3)$$

For the sake of convenience to analyze the tunneling radiation characteristics, using the parameters to replace m, α, β, μ with M, Q, J,

$$m = M - \frac{Q^2}{2M}, \quad \beta = -\frac{Q^2}{2M}, \quad a = \frac{J}{M}, \quad \mu = aQ. \quad (4)$$

From null surface equation

$$g^{\mu\upsilon}\frac{\partial f}{\partial x^{\mu}}\frac{\partial f}{\partial x^{\upsilon}}=0, \tag{5}$$

we get the outer and inner event horizons

$$r_{\pm}=m\pm\sqrt{m^2-a^2}=M-\frac{Q^2}{2M}\pm\sqrt{\left(M-\frac{Q^2}{2M}\right)^2-a^2}. \tag{6}$$

For the sake of convenience, we firstly calculate the area of the black hole. When t_s is a constant, in case of $r=r_+$, the line element (1) can be written as follows

$$d\sigma^2=\Delta d\theta^2+\frac{\sin^2\theta}{\Delta}\left[(r^2+a^2-2\beta r)^2-\sum a^2\sin^2\theta\right]d\varphi^2, \tag{7}$$

The determinant of the two dimensional metric is

$$g=\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix}=\sin^2\theta(r_+^2+a^2-2\beta r_+)^2, \tag{8}$$

So the area of the black hole is

$$\begin{aligned} A&=\int dA=\int\sqrt{g}d\theta d\varphi=4\pi(r_+^2+a^2-2\beta r_+)\\ &=8\pi M\left[M-\frac{Q^2}{2M}+\sqrt{(M-\frac{Q^2}{2M})^2-a^2}\right]. \end{aligned} \tag{9}$$

From $g_{00}=-\frac{\sum-a^2\sin^2\theta}{\Delta}=0$, we can get infinite red-shift surface

$$r_{\pm}^{s}=m\pm\sqrt{m^2-a^2\cos^2\theta}. \tag{10}$$

Obviously the infinite red-shift surface doesn't coincide with event horizon. Making dragging coordinate transformation and noting

$$\dot{\varphi}=\frac{d\varphi}{dt_S}=-\frac{g_{03}}{g_{33}}, \tag{11}$$

we can obtain

$$ds^2=\hat{g}_{00}dt_S^2+\frac{\Delta}{\sum}dr^2+\Delta d\theta^2, \tag{12}$$

where

$$\hat{g}_{00}=g_{00}-\frac{g_{03}^2}{g_{33}}=-\frac{\sum\Delta}{(r^2-2\beta r+a^2)^2-\sum a^2\sin^2\theta}, \tag{13}$$

Then, when $\hat{g}_{00}=0$, the infinite red-shift surface is

$$r_{\pm}^{irs}=m\pm\sqrt{m^2-a^2}. \tag{14}$$

Comparing with (6), we get that the infinite redshift surface coincide with the event horizon in dragging coordinate system.

3. GENERAL PAINLEVE COORDINATE TRANSFORMATION

It is necessary to eliminate coordinate singularity to analyze the Hawking radiation effect as tunneling at the event horizon of the black hole. In (12), there still exists coordinate singularity in dragging coordinate system. Accordingly we further make general Painleve coordinate transformation (Painleve and Hebd, 1921; Zhang and Zhao, 2005) and let

$$dt_S = dt + F(r,\theta)\,dr + G(r,\theta)\,d\theta, \tag{15}$$

in which, the integrability conditions is

$$\frac{\partial F(r,\theta)}{\partial\theta} = \frac{\partial G(r,\theta)}{\partial r}. \tag{16}$$

Substituting (15) into (12), we have

$$\begin{aligned} ds^2 = {} & \hat{g}_{00}dt^2 + \left(\hat{g}_{00}F^2(r,\theta) + \Delta\sum\nolimits^{-1}\right)dr^2 + (\hat{g}_{00}G^2(r,\theta) + \Delta)d\theta^2 \\ & + 2\hat{g}_{00}F(r,\theta)\,G(r,\theta)\,drd\theta + 2\hat{g}_{00}F(r,\theta)\,dtdr \\ & + 2\hat{g}_{00}G(r,\theta)\,dtd\theta, \end{aligned} \tag{17}$$

Considering flat Euclidean space in radial, we get

$$\hat{g}_{00}F^2(r,\theta) + \Delta\sum\nolimits^{-1} = 1, \tag{18}$$

namely

$$F(r,\theta) = \pm\sqrt{\frac{1-\Delta\sum^{-1}}{\hat{g}_{00}}}, \tag{19}$$

from which, we can get the space-time line element in Painleve coordinate

$$\begin{aligned} ds^2 = {} & \hat{g}_{00}dt^2 + dr^2 + 2\sqrt{\hat{g}_{00}\left(1-\Delta\sum\nolimits^{-1}\right)}dtdr + [\hat{g}_{00}G^2(r,\theta) + \Delta]d\theta^2 \\ & + 2\hat{g}_{00}G(r,\theta)\,dtd\theta + 2\sqrt{\hat{g}_{00}\left(1-\Delta\sum\nolimits^{-1}\right)}G(r,\theta)\,drd\theta. \end{aligned} \tag{20}$$

According to Landau's condition of coordinate clock synchronization (Landau and Lifshitz, 1975)

$$\frac{\partial}{\partial x^i}\left(-\frac{g_{0j}}{\hat{g}_{00}}\right) = \frac{\partial}{\partial x^j}\left(-\frac{g_{0i}}{\hat{g}_{00}}\right), \tag{21}$$

we obtain

$$\frac{\partial F(r,\theta)}{\partial \theta} = \frac{\partial G(r,\theta)}{\partial r}. \tag{22}$$

which is equivalent to Eq. (16), then we can infer that the space-time line element in new coordinate satisfies the Landau's condition of coordinate clock synchronization. This is the essential element to study the tunneling effect of the black hole. Moreover, according to (20), the space-time line element in Painleve coordinate possesses various superior features, such as without singularity at event horizon, coincidence between its event horizon and infinite red-shift surface, and the line element is flat Euclidean space in radial. All these provide superior conditions to study the quantum tunneling radiation of black hole.

4. THE TUNNELING RADIATION CHARACTERISTICS OF SEN BLACK HOLE

We consider uncharged massless particle tunneling radiation coming from the black hole. From (20), the radial null geodesics equations are

$$\begin{aligned}\dot{r} = \frac{dr}{dt} &= -\sqrt{\hat{g}_{00}\left(1 - \Delta \sum^{-1}\right)} \pm \sqrt{-\hat{g}_{00}\Delta \sum^{-1}} \\ &= \frac{\pm\Delta - \sqrt{\Delta\left(\Delta - \sum\right)}}{\sqrt{(r^2 + a^2 - 2\beta r)^2 - \sum a^2 \sin^2\theta}},\end{aligned} \tag{23}$$

where the $\pm$ signs correspond to outgoing and ingoing geodesics. Consider that a pair of virtual particles spontaneously creates just inside the horizon, the positive energy virtual particle can tunnel out and the negative energy particle is absorbed by the black hole. Under considering self-gravitating action, energy conservation and angular momentum conservation, the particle is as a shell (an ellipsoid shell) of energy ω' and angular momentum $\omega' a$. When a particle tunnels out, the black hole's mass will become $M - \omega'$ and the angular momentum of the black hole will become $(M - \omega')a$. Meanwhile, the event horizon will shrink, to event horizons responding to the cases pre- and post shrinking are two turning points of potential barrier. The distance between the two barriers is the width of potential barrier and dependent on the energy of outgoing particles. Considering energy conservation and angular momentum conservation during the particles' turning process, Thus the line element (20) and Eq. (23) should replace M with $(M - \omega')$, and the event horizon and angular speed are respectively

$$r'_+ = (M-\omega) - \frac{Q^2}{2(M-\omega)} + \sqrt{\left((M-\omega) - \frac{Q^2}{2(M-\omega)}\right)^2 - a^2},$$

$$\dot{\varphi} = \frac{d\varphi}{dt} = -\frac{g_{03}}{g_{33}}\bigg|_{r'_+} = \Omega'_H = \frac{a}{a^2 + {r'_+}^2 + \frac{Q^2}{(M-\omega)} r'_+}, \tag{24}$$

Since the outer event horizon coincides with outer infinite red-shift surface, geometrical optics limit exists. According to WKB approximation tunneling rate and the action of the particle satisfy (Kraus and Keski-Vakkuri, 1997; Kraus and Parentani, 2000)

$$\Gamma \sim e^{-2\mathrm{Im}S}. \tag{25}$$

The imaginary part of particle's action is

$$\begin{aligned} ImS &= Im\int_{t_i}^{t_f} (L - P_\varphi \dot{\varphi})dt = Im\left[\int_{r_i}^{r_f} P_r dr - \int_{\varphi_i}^{\varphi_f} P_\varphi d\varphi\right] \\ &= Im\left[\int_{r_i}^{r_f}\int_0^{p_r} dP'_r - \int_{\varphi_i}^{\varphi_f}\int_0^{p_\varphi} dP'_\varphi d\varphi\right] dr. \end{aligned} \tag{26}$$

in which φ as an ignorable coordinate in the Lagrange function is considered and is just the expression of action after amendments. Take Hamilton equation into account, then

$$\begin{aligned} \dot{r} &= \frac{dH}{dP_r}\bigg|_{(r;\varphi,P_\varphi)} = \frac{d(M-\omega')}{dp_r}\dot{\varphi} = \frac{dH}{dP_\varphi}\bigg|_{(\varphi;r,P_r)}, \\ dH_{(\varphi;r,P_r)} &= \Omega'_H dJ = \Omega'_H a d(M-\omega') \end{aligned} \tag{27}$$

so

$$ImS = Im\int_0^{\omega}\int_{r_i}^{r_f} -(1 - a\Omega'_H)\frac{d\omega'}{\dot{r}} dr. \tag{28}$$

Take integration on r, then

$$\begin{aligned} &\int_{r_i}^{r_f} -(1 - a\Omega'_H)\frac{dr}{\dot{r}} \\ &= -\int_{r_i}^{r_f}\left(\frac{{r'_+}^2 - 2\beta' r'_+}{a^2 + {r'_+}^2 - 2\beta' r'_+}\right) \\ &\quad \times \frac{\left[\Delta + \sqrt{\Delta\left(\Delta - \sum\right)}\right]\sqrt{\left(r^2 + a^2 - 2\beta' r\right)^2 - \sum a^2 \sin^2\theta}}{\Delta(r - r'_+)(r - r'_-)} dr \\ &= -\int_{r_i}^{r_f}\left(\frac{{r'_+}^2 - 2\beta' r'_+}{a^2 + {r'_+}^2 - 2\beta' r'_+}\right)\frac{\left[\Delta + \sqrt{\Delta\left(\Delta - \sum\right)}\right]\sqrt{\left(r^2 + a^2 - 2\beta' r\right)^2 - \sum a^2 \sin^2\theta}}{\Delta(r - r'_+)(r - r'_-)} dr \\ &= 2\pi i\frac{{r'_+}^2 - 2\beta' r'_+}{r'_+ - r'_-}. \end{aligned} \tag{29}$$

where $\beta' = -\frac{Q^2}{2(M-\omega)}$, Then carrying on integrality on ω, we obtain.

$$
\begin{aligned}
&ImS \\
&= Im\int_0^{\omega} 2\pi i\frac{{r'_+}^2 - 2\beta' r'_+}{r'_+ - r'_-}d\omega' \\
&= 2\pi i\int_0^{\omega} \frac{2(M-\omega')^2 - Q^2 - a^2 + 2(M-\omega')\sqrt{\left((M-\omega') - \frac{Q^2}{2(M-\omega')}\right)^2 - a^2}}{2\sqrt{\left((M-\omega') - \frac{Q^2}{2(M-\omega')}\right)^2 - a^2}}d\omega \\
&= \pi\left[M^2 - (M-\omega)^2 + \sqrt{M^4 - (a^2+Q^4)M^2 + \frac{Q^4}{4}}\right. \\
&\quad \left. - \sqrt{(M-\omega)^4 - (a^2+Q^4)(M-\omega)^2 + \frac{Q^4}{4}}\right].
\end{aligned} \tag{30}
$$

The tunneling rate of outgoing particle tunnel out a radial barrier is

$$
\begin{aligned}
\Gamma &\sim e^{-2ImS} \\
&= \exp\left\{-2\pi\left[M^2 - \left(M-\omega\right)^2 + \sqrt{M^4 - (a^2+Q^4)M^2 + \frac{Q^4}{4}}\right.\right. \\
&\quad \left.\left. - \sqrt{(M-\omega)^4 - (a^2+Q^4)(M-\omega)^2 + \frac{Q^4}{4}}\right]\right\}
\end{aligned} \tag{31}
$$

Namely we have

$$
\Gamma \sim e^{-2ImS} = e^{\frac{A'}{4} - \frac{A}{4}} = e^{S_{BH}(M-\omega) - S_{BH}(M)} = e^{\Delta S_{BH}}, \tag{32}
$$

where

$$
A' = 8\pi\,(M-\omega)\left[(M-\omega) - \frac{Q^2}{2\,(M-\omega)} + \sqrt{\left((M-\omega) - \frac{Q^2}{2\,(M-\omega)}\right)^2 - a^2}\right],
$$

A' is the area of the Sen black hole after radiation and S_{BH} is Bekenstein–Hawking entropy.

5. DISCUSSION

Stationary axisymmetric Sen black hole exists magnetic monopoles, which is different from Kerr–Newman black hole. Then the tunneling rate of particles at the event horizon is different from that of Kerr–Newman black hole. The solution of revolving uncharged black hole is identical to the one of Kerr. When $Q = 0$, the stationary axisymmetric Sen black hole transforms back into Kerr black hole. According to (31), we have

$$\Gamma \sim e^{-2ImS} = e^{-2\pi\left[M^2-(M-\omega)^2+M\sqrt{M^2-a^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2}\right]} = e^{\Delta S_{BH}}, \qquad (33)$$

which is just the tunneling rate of Kerr black hole (Zhang and Zhao, 2005). When $Q = 0$, $\mu = 0$ and $a = 0$,

$$\Gamma \sim e^{-2ImS} = e^{-8\pi\omega\left(M-\frac{\omega}{2}\right)} = e^{\Delta S_{BH}}. \qquad (34)$$

this is the tunneling rate of Schwarzschild, which is concordance with the previous one (Parikh and Wiltzek, 2000). Taking all the above research into account, black hole radiation causes the space-time background geometry varied because of self-gravitation and energy conservation and angular momentum conservation, the event horizon of black hole varies with black hole radiation, namely when the particle outgoes, the event horizon will contract and the two turning points pre-contraction and post contraction are the two points of barrier. The tunneling rate of particle is relevant to M, a, Q, μ, and when $Q = a = \mu = 0$, the tunneling is equal to the quantum tunneling rate of Schwarzschild black hole. And only ignoring item ω^2 concords with the pure thermal spectrum of Hawking radiation in Schwarzschild black hole. The characteristics of stationary axisymmetric Sen black hole is researched through above methods, in which the exact expression of tunneling rate is obtained and the result is as a good amendment to Hawking pure thermal spectrum.

ACKNOWLEDGMENTS

Great thanks are due to Prof. Zhang Jingyi and Doctor Ren Jun for their helpful discussion. The work was supported by the National Foundation of China under Grant N0.10347008.

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